GEOMETRICAL THEORY
OF THE
DETERMINATION OF PRICES.*

Translator's Note.

The following article of Professor Walras is the first by that famous writer to appear in English. It is not too much to say that Professor Walras, together with Jevons and Menger, marked an epoch in political economy. These three writers independently and almost simultaneously developed and applied the idea of *rareté* or marginal utility, the corner-stone of the so-called "Austrian School" and of mathematical economics. Those who are not already familiar with the services which the *Économie Politique pure* has rendered will be interested to read the praise of Jevons (Theory of Pol. Econ. 3d Ed. Preface to 2d Ed. p. xxxviii.).

The problem which Professor Walras has here set himself is a difficult one, viz.: to present a geometric picture of the causation of the prices of *all* commodities, recognizing the fact that these prices are mutually dependent. I have never seen it attempted elsewhere. In his *Économie Politique pure* he treated this problem analytically and was, I feel confident, the first to do so. Auspitz and Lieben also in their much praised "*Untersuchungen über die Theorie des Preises*" use their ingenious diagrams only for a single commodity and content themselves with an analytical treatment as soon as they pass to several commodities. The present writer is about to publish in the transactions of the Connecticut Academy of Arts and Sciences a *Mathematical Theory of Value and Prices*, in which is a solution of this same problem, by the aid of an instrument quite different from either the analytic or graphical method. An actual mechanism is constructed so as to exhibit the complex market adjustment as an automatic equilibrium of a system of liquids.

* Of the three parts of which this paper is composed, the first consists of a memoir read before the Society of Civil Engineers of Paris, October 17, 1890, printed in the *Bulletin* of that Society, January, 1891. In it the author has, however, made certain modifications, of which one is rather important, because it simplifies the fundamental demonstration of the theorem of maximum satisfaction. The last two consist, with certain modifications necessitated by what preceded, of a paper prepared for the *Recueil inaugural* of the University of Lausanne. The editors of the *Recueil* kindly authorized the transmission of the MS. of this article, previous to its publication in Lausanne, to the *American Academy of Political and Social Science*, with a view to insertion in its ANNALS.

A few words of comment on the present article may be appropriate:

(1) Although Professor Walras clearly recognizes that the demand and supply of each commodity is a function of the prices of all commodities, he omits to state that the *rareté* to a given individual of a given amount of one commodity is a function of the quantities not only of that commodity, but of all others. Hence the curves he employs are not independent, but the shape of the A curve in Fig. 1 will change according to what point is selected on the B curve. The A curve could not be said to be given until the demand for B, C, D, etc., was each given. The utility of bread, it is true, decreases with the amount of bread, but the *law* of that decrease depends on the amount of butter; in general the utility of the same quantity of bread increases as the amount of butter increases.

(2) With a slight change of phraseology the methods and curves used by Professor Walras would apply to the *rate* of consumption in time. Thus the quantities Oq_a, Oq_b, Oq_c, etc. (Fig. 1), could be taken to mean the amounts of each commodity *annually* produced by the given individual and disposable for consumption and exchange. The vertical length, used in Fig. 2, $q_a + q_b p_b + q_c p_c +$ etc., would then indicate his annual income measured in terms of the commodity A. I am aware that many economists object to such an introduction of the time element, but their objections disappear as soon as the notion of a statical market is admitted. To limit price analysis to a single instant and the supply to the amounts of stock then on the market, unnecessarily restricts its range of application, and "dealing in futures" makes its meaning vague. What can be meant by the amount of petroleum on the New York market at an instant? Many gallons are flowing through pipes and in twenty-four hours more than a million gallons will be added. The *rate* of production and not the stock is the well defined quantity, and so with all staple articles.

(3) Professor Walras goes unnecessarily out of his way in making his special supposition that in order to have a common cost price equal to the selling price suppliers must produce equal quantities (by which presumably is meant equal quantities per unit of time). The rate of production regulates the cost of production, that is, the *marginal* cost or sacrifice, and it is quite possible for this marginal cost (measured in money) to be the same for a small cobbler as for a large shoe manufacturer, though the fixed and running expenses may divide themselves very differently. Moreover, it is rather misleading to say that the equality of [marginal] cost and selling price implies neither gain nor loss. There exists a normal gain or "producer's rent," as Marshall calls it, which is quite distinct from and inde-

pendent of any speculative gains or losses, $\Omega_b\,(\pi''_b - p_b)$ etc., due to a disturbance of the equilibrium between cost and selling price.

IRVING FISHER.

I.

THE EXCHANGE OF SEVERAL COMMODITIES AMONG THEMSELVES.

In my *Eléments d'économie politique pure*,* passing from the theory of the exchange of two commodities to the theory of the exchange of several commodities among themselves, and seeing that in that case the demand or the supply of each of the commodities by each of the traders is a function, not only of the price of that commodity, but also of the price of all the others, I believed it was necessary to adopt exclusively the analytic method of expression and do without the help of diagrams. But since then I have found a means, which I will indicate briefly, of elaborating the theory in question by the method of geometrical representation.

Suppose a party to the exchange with the quantities q_a, q_b, q_c, q_d . . . of the commodities (A), (B), (C), (D), represented by the lines Oq_a, Oq_b, Oq_c, Oq_d . . . (Fig. 1) and having for him the utility expressed by the curves $\alpha_q\,\alpha_r$, $\beta_q\,\beta_r$, $\gamma_q\,\gamma_r$, $\delta_q\,\delta_r$ . . . I proceed to describe these curves which are the essential and fundamental basis of all the mathematical theory of social wealth.

We may say in ordinary language: "The desire that we have of things or the utility that things have for us, diminishes in proportion to the consumption. The more a man eats, the less hungry he is; the more he drinks, the less thirsty; at least in general and saving certain regrettable exceptions. The more hats and shoes a man has, the less need has he of a new hat or a new pair of shoes; the more horses he has in his stables, the less effort will he make to procure one horse more, always neglecting the action of impulses which the theory has the right to neglect, excepting when accounting for certain special cases." But in mathematical terms we say: "The intensity of the last desire satisfied, is

* Lausanne. F. Rouge. 1889.

a decreasing function of the quantity of commodity consumed," and we represent these functions by curves, the *quantities consumed* by the ordinates and the *intensity of the last desire satisfied* by the abscissas. For example, take the commodity (A), the intensity of the desire of our consumer which would be $O\alpha_r$ at the beginning of the consumption, would be nil after the consumption of a quantity $O\alpha_q$, the consumer having then arrived at satiety. That intensity of the last desire satisfied, for the sake of brevity, I call *rareté*. The English call it the *final degree of utility*, the Germans *Grenznutzen*. It is not an appreciable quantity, but it is only necessary to conceive it in order to found upon the fact of its diminution the demonstration of the great laws of pure political economy.

For the present let $p_b, p_c, p_d \ldots$ be the prices of (B), (C), (D), in terms of (A) determined at random on the market. The first problem that we have to solve consists in determining the quantities of (A), (B), (C), (D) . . ., $x, y, z, w \ldots$ the first positive and representing the quantities demanded, the second negative and representing the quantities offered, which our trader will add to the quantities q_a, q_b, q_c, q_d, of which he is already possessed or which he will subtract from them, so as to consume the quantities $q_a + x$, $q_b + y$, $q_c + z$, $q_d + w \ldots$ represented by the lines Oa, Ob, Oc, $Od \ldots$ Just as we employed the general hypothesis above, of a party to the exchange for whom the *rareté* decreased with the quantity consumed, so here we employ the general hypothesis of a party to the exchange who seeks in the exchange the greatest possible satisfaction of his desires. Now the sum of the desires satisfied by a quantity Oa of commodity (A), for example, is the *surface* $Oa\rho_a\alpha_r$. The *effective utility* is the integral described by the *rareté* in relation to the quantity consumed. Consequently the problem, whose solution we are seeking, consists precisely in determining Oa, Ob, Oc, $Od \ldots$ under the condition that the sum of the shaded areas $Oa\rho_a\alpha_r$, $Ob\rho_b\beta_r$, $Oc\rho_c\gamma_r$, $Od\rho_d\delta_r \ldots$ be a maximum.

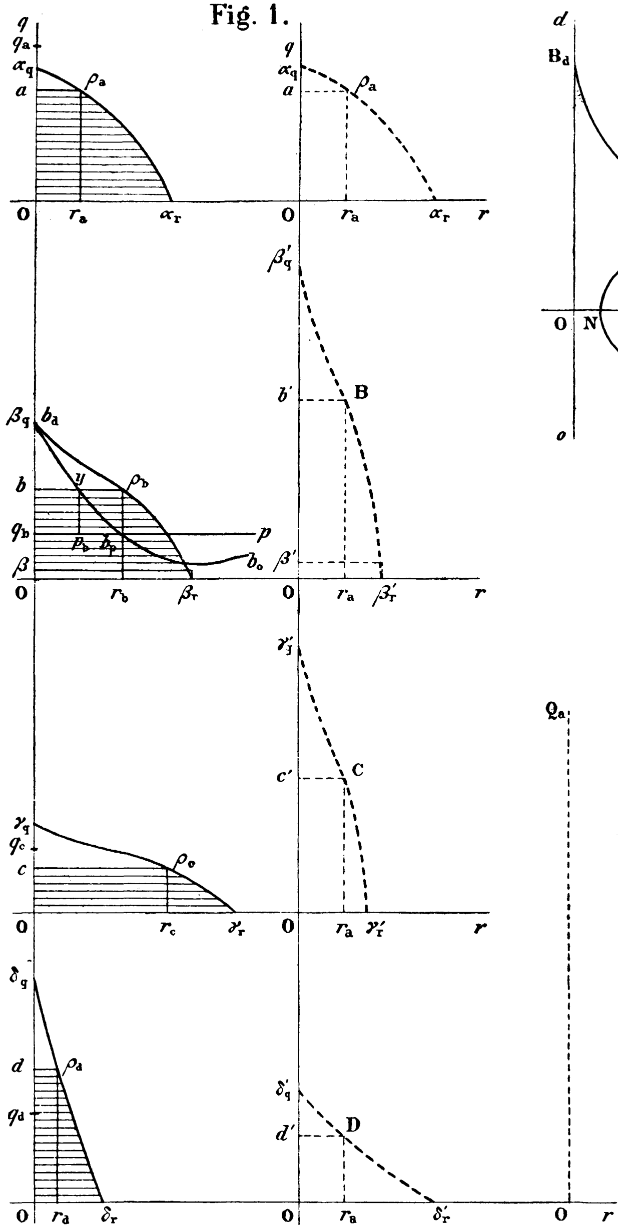

Fig. 1.

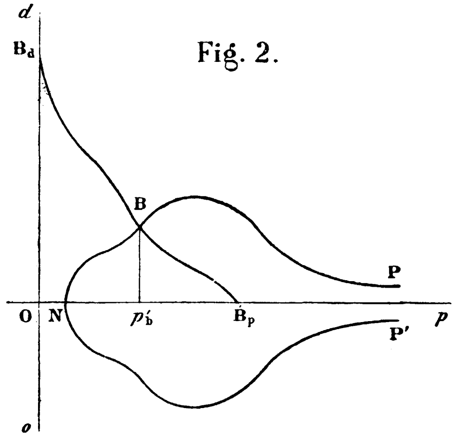

Fig. 2.

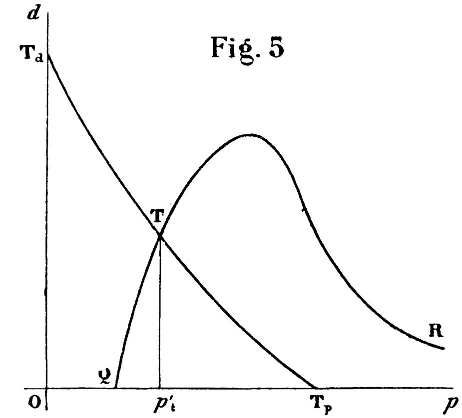

Fig. 5

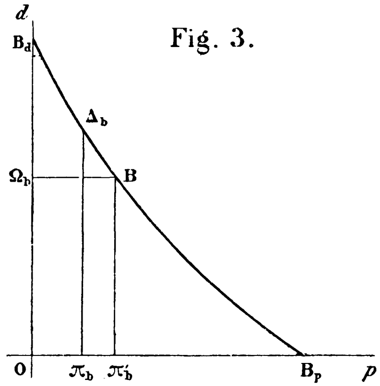

Fig. 3.

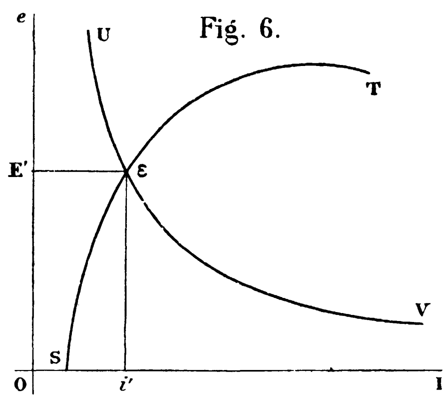

Fig. 6.

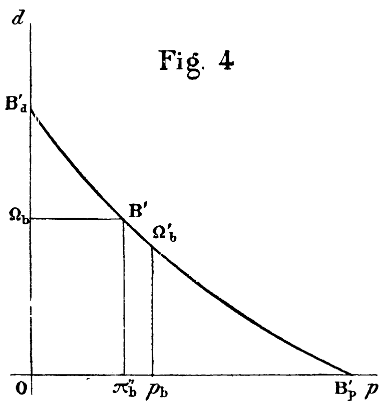

Fig. 4

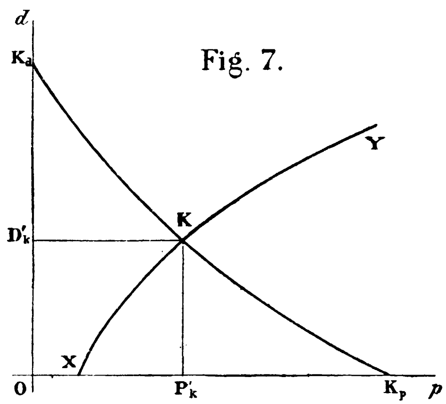

Fig. 7.

L. Walras.

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In order to furnish that solution very simply in the geometric form, I subject the curves of utility or desire $\beta_q \beta_r$, $\gamma_q \gamma_r$, $\delta_q \delta_r$ . . . to the following transformation. I lay off from the origin O, upon the horizontal axes, the new abscissas equal to $\frac{1}{p}$ of the old abscissas. Also, upon the parallels to the vertical axes drawn through the extremities of the new abscissas, I lay off from the horizontal axes the new ordinates equal to p times the old ordinates. In the figure, $p_b = 2$, $p_c = 3$, $p_d = \frac{1}{2}$ . . . As is easily seen, the new curves $\beta'_q \beta'_r$, $\gamma'_q \gamma'_r$, $\delta'_q \delta'_r$ . . . represent the utility of (A), as spent for (B), for (C), for (D) . . ., or, in other words, the desire the exchanging party has of (A), in order to procure some of (B), of (C), of (D). . . In short, if we consider the areas $O\beta_q\beta_r$, $O\gamma_q\gamma_r$, $O\delta_q\delta_r$ . . . as the limits of sums of rectangles infinitely small, we may consider the surfaces, $O\beta'_q \beta'_r$, $O\gamma'_q \gamma'_r$ $O\delta'_q \delta'_r$, as the limits of *equal* sums of rectangles infinitely small, each base being p times less, and each height p times greater. Now, each of the rectangles of the former sum represents the effective utility of an increment of commodity; each of the rectangles of the latter sum represents, in the same way, the equal effective utility of the p increments of (A), with which that increment of commodity may be bought.

The curves $\alpha_q\alpha_r$, $\beta'_q\beta'_r$, $\gamma'_q\gamma'_r$, $\delta'_q\delta'_r$ being placed beside each other, I take a vertical length OQ_a, representing the equivalent in (A) of the quantities q_a, q_b, q_c, q_d, of (A), (B), (C), (D) . . . at the prices 1, p_b, p_c, p_d . . . viz: $q_a + q_b p_b + q_c p_c + q_d p_d +$ . . . and I advance it from right to left, in order to satisfy the varying desires in the order of their intensity, until it is sub-divided among the curves into the ordinates $r_a \rho_a = Oa$, $r_a B = Ob'$, $r_a C = Oc'$, $r_a D = Od'$, . . . corresponding to a like abscissa. Now, that abscissa Or_a will represent, in terms of (A), the *rareté* of (A), of (B), of (C), of (D) . . . say, r_a, corresponding to the maximum of effective utility. The ordinates Oa, Ob', Oc', Od' . . ., will represent, in terms

of (A), the quantities to be consumed, of (A), of (B), of (C), and of (D), the only commodities to be consumed being those for which the intensity of the first desire to be satisfied is greater than r_a.

If we carry back the abscissas $Or_a = r_a$, $Or_b = p_b r_a$, $Or_c = p_c r_a$, $Or_d = p_d r_a$, . . . to the curves $\alpha_q\, \alpha_r$, $\beta_q\, \beta_r$, $\gamma_q\, \gamma_r$, $\delta_q\, \delta_r$ . . . we obtain the ordinates Oa, Ob, Oc, Od . . . representing the quantities of (A), of (B), of (C), of (D), . . . to be consumed. And so, *in a state of maximum satisfaction, the raretés are proportional to the prices, according to the equations:*

$$\frac{r_a}{1} = \frac{r_b}{p_b} = \frac{r_c}{p_c} = \frac{r_d}{p_d} = \ \ldots$$

Thus it is, that, given the quantities possessed and the utilities of the commodities, we determine for a party in the exchange the demand or supply of each of the commodities at prices taken at random, which will afford the maximum satisfaction of his wants.

Having given the demand and supply of commodities by all the parties in the exchange at prices taken at random, it remains to determine the current prices at equilibrium, under the condition of the equality of the total effective demand and supply. The solution of the second problem may also be furnished geometrically.

Let us, for an instant, neglect p_c, p_d . . . and seek at first to determine, provisionally, p_b. And, for that purpose, let us inquire how (p_c, p_d ... being supposed constant) the variations of p_b influence the demand and supply of (B).

If y is positive, that is to say, if the trader is in need of (B), an augmentation of p_b can only diminish y. In short, if he takes at a higher price an equal quantity, he still owes a difference which he cannot pay, without diminishing the quantities of (A), (C), (D) . . . But, then, he will augment the *raretés* of these commodities; and, in consequence, the condition of maximum satisfaction will be less perfectly fulfilled. Hence, the quantity y is too great for a price higher than p_b.

If y is negative, that is to say, if the party is a supplier of

(B), there are three possible results. The party, being supposed to supply an equal quantity at a higher price, a surplus is due him, and, by means of that surplus, he can augment his quantities, and consequently diminish his *raretés* of (A), (C), (D). . . Then, one of three things occurs: Either the surplus is insufficient to re-establish the condition of maximum satisfaction, or it is just sufficient, or it is more than sufficient; and, in consequence, at a price higher than p_b, the party must supply a quantity of (B), either greater than, equal to, or less than y. It is certain, that he will find himself in one of these three cases, according to the amount of the enhancement of p_b. For, if, on the one hand, that enhancement of p_b constantly diminishes the ratio $\frac{r_b}{p_b}$ by increasing the denominator, that same increase of p_b admits, on the other hand, of a continual lowering in $\frac{r_a}{1}, \frac{r_c}{p_c}, \frac{r_d}{p_d} \ldots$ by a diminution of the numerators $r_a, r_c, r_d, \ldots$ which may finally cause a decrease of the numerator r_b itself.

The variation of p_b, from zero to infinity, therefore, causes the party to the exchange to pass from the side of demand to that of supply; then, from an increasing supply to a decreasing supply. At the price zero, the demand is equal to the excess of the quantity necessary perfectly to satisfy the wants over the quantity possessed; at the price infinity, the amount offered is nil. In the case of the exchange of several commodities, as in the case of the exchange of two commodities with each other, the tendencies may be represented geometrically, for a party to the exchange, by a curve* $b_d\, b_p\, b_o$. (Fig. 1).

* [The ordinates of this curve have the same meaning as the ordinates of the utility curve for (B) on which it is superposed, but its abscissas represent the price of (B) in terms of (A.) Thus in order that the individual may consume Ob, that is buy $q_b\, b$ in addition to his original stock Oq_b the price must be $b\, y$ (in this case 2). If the price rises above $q_b\, b_p$ he ceases to buy and begins to sell. But the curve

All the parties to the exchange being not identical, but similar in their tendencies, as far as concerns the commodity (B), it is clear that all the partial curves of demand must be united† in a total curve that continually decreases, $B_d\,B_p$ (Fig. 2), and all the partial curves of supply in a total curve NP, successively increasing and decreasing, from zero to zero, if we take it positively, by making NP′ turn around the horizontal axis, so as to bring it to the position NP. The abscissa Op'_b, of the point of intersection B of the two curves $B_d\,B_p$ and NP, will be provisionally the current price at equilibrium for which the total effective supply and demand of (B) will be equal. Furthermore, the intersection of the two curves, $B_d\,B_p$ and NP, may take place either when the second curve rises, or when it falls.

It follows from the nature of the curves, that we shall obtain the provisional current price of (B) by raising it in case of a surplus of effective demand over effective supply, and lowering it, on the contrary, in case of a surplus of effective supply over effective demand. Passing then to the determination of the current price of (C), then to the current price of (D) . . ., we obtain them by the same means. It is quite true that, in determining the price of (C), we may destroy the equilibrium in respect to (B); that, in determining the price of (D), we may destroy the equilibrium in respect to (B), and in respect to (C), and so on. But, as the determi-

reaches a minimum point and then approaches $q_b\,p$ as an asymptote. That is as the price rises beyond a certain point, he ceases to stint himself in the enjoyment of (B) and parts with less and less of it. —TRANSLATOR.]

† [The total *demand* curve for (B) ($B_d\ B_p$ Fig. 2) may evidently be found from the partial curves such as $b_d\ b_p\ b_o$ by selecting on these individual curves the points which correspond to a given abscissa (price) and constructing a corresponding point in the total curve which shall have the same abscissa, but whose ordinate shall be the sum of the ordinates of the individual curves *measured above* $q_b\ p$ as $y p_b$. In like manner the ordinates of NP′ are the sum of the individual ordinates *measured below* $q_b\,p$ and corresponding to like abscissas.—TRANSLATOR.]

nations of the prices of (C), (D) . . . in respect to the demand and supply of (B), will result in a contrary way, we shall always be nearer the equilibrium at the second trial than at the first. We enter here on the theory of trial and error, such as I have developed in my work, and by virtue of which *we arrive at the equilibrium of a market by raising the price of commodities, the demand for which is greater than the supply, and by lowering the price of those, the supply of which is greater than the demand.*

It is due to the concurrent employment of analytic expression and geometric representation that we have here, in the case of the exchange of several commodities among themselves, not only the idea but the picture of the phenomenon of the determination of prices upon the market. And with this, it seems to me, we possess at last the theory. Some critics, however, laugh at the number of pages I use in demonstrating that we may arrive at a current price by raising in case of an excess of the demand over the supply, and lowering in case of an excess of the supply over the demand. "And you," I said once to one of them, "how do you demonstrate it?" "Well," he answered me, a little surprised and embarrassed, "is there any need of demonstrating it? It seems to me self-evident." "There is nothing evident except axioms, and this is not an axiom. But one naturally follows the mode of reasoning which Jevons has formulated so clearly in his little treatise on *Political Economy*, that a rise, making necessarily a diminution of the demand and an augmentation of the supply, causes equality in case of a surplus of the one over the other." "Precisely." "But there is an error there. A rise necessarily diminishes the demand; but it does not necessarily augment the supply. If you are a supplier of wine, it may well be that you supply less at a million, than at a thousand francs, less at a billion than at a million, simply because you prefer to drink your wine yourself, rather than use the surplus which you could procure by selling it beyond a certain limit. The same is true of labor. We easily

conceive that a man, who supplies ten hours a day of his time at the price of one franc an hour, would not supply more than four at the price of 10 francs, or than one at 100 francs. We see, every day, in the large towns, that the laborers, when they earn 20 or 25 francs a day, do not work more than three or four days a week." "But if that is so, how is raising it a means of reaching the current price?" "It is this that the theory explains. Two individuals, who have separated, may meet again, either by moving each, in an opposite direction to the other, or by one going faster than the other. Supply and demand equalize themselves, sometimes in one way, sometimes in another." Is it not worth while to demonstrate rigorously the fundamental laws of a science? We count to-day I do not know how many schools of political economy. The *deductive* school and the *historical* school; the school of *laisser-faire* and the school of *state-intervention*, or *socialisme de la chaire*, the *socialistic* school properly so-called, the *catholic* school, the *protestant* school. For me, I recognize but two: the school of those who do not demonstrate, and the school, which I hope to see founded, of those who do demonstrate their conclusions. It is in demonstrating rigorously the elementary theorems of geometry and algebra, then the theorems of the calculus and mechanics which result from them, in order to apply them to experimental ideas, that we realize the marvels of modern industry. Let us proceed in the same way in political economy, and we shall, without doubt, succeed in dealing with the nature of things in the economic and social order, as they are dealt with in the physical and industrial order.

II.

THE EXCHANGE OF PRODUCTS AND SERVICES WITH EACH OTHER.

It is my present purpose to apply to the theory of production and the theory of capitalization the exclusively geometric method of demonstration according to which I have sketched the theory of exchange in the preceding paragraph.

Now, in formulating the theory of exchange, we suppose the quantities of the commodities to be a given, not an unknown element of the problem. To begin with, in order to arrive at a theory of production, it is necessary to consider commodities as the products resulting from combined productive services, and, in consequence, it is necessary to introduce the quantities of manufactured products into the problem, as so many unknown quantities, adding as is proper, an equal number of determining mathematical conditions. That is what I wish to do here, referring to my *Elements of pure political economy* for definitions and notations.*

Suppose, then, the services of land, labor and capital [in the narrow sense] (T),(P), (K) . . . susceptible of being utilized, either directly as consumable services, or indirectly as productive services, that is to say, in the form of the products of the sorts (A), (B), (C), (D) . . . The first problem that we have to solve consists in determining, for each consumer, the supply of services and the demand for services, either in the form of consumable services, or as products. Now the solution of the problem is furnished us by the theory of exchange.

Given, then, a consumer possessed of the quantities q_t, q_p, q_k, of the services (T), (P), (K) . . . and having a desire for these services and a desire for the products, (A), (B), (C), (D) . . . expressed by the curves of utility or desire. Given, also, p_t, p_p, p_k . . . π_b, π_c, π_d, . . . the prices (taken at random) of (T), (P), (K) . . . and of (B), (C), (D) . . . in terms of (A). We will transform the curves of utility or desire of services and products into curves of utility (measured in (A)) of (T), (P), (K) . . . (B), (C), (D) . . . or, in other words, into curves of the desire for (A) to be used in procuring some of (T), (P),

* [These definitions of conceptions peculiar to the author and constituting essential elements of his system of Economics are to be found in the preface to his work (pp. XII–XVI). I have translated by *capitals* the word *capitaux* (which includes all things material or immaterial that are used more than once) and by *services* the word *services* (the successive uses of the capitals).—TRANSLATOR.]

(K), . . . (B), (C), (D) . . . This is done by dividing the abscissas and multiplying the ordinates by market prices. The curve of the utility or desire of (A) and the transformed curves of utility, or desire of (T), (P), (K) . . . (B), (C), (D) . . . being placed one under the other, we may advance a vertical line of the length $Q_a = q_t p_t + q_p p_p + q_k p_k + \ldots$ from right to left, until it distributes itself among all the curves into ordinates corresponding to a like abscissa r_a. By carrying back the abscissas $p_t r_a$, $p_p r_a$, $p_k r_a \ldots r_a$, $\pi_b r_a$ $\pi_c r_a$, $\pi_d r_a \ldots$ to the primitive curves, we obtain the ordinates representing the quantities of labor (T), (P), (K) . . . and of products (A), (B), (C), (D) . . . to be consumed. It is evident that *in the state of maximum satisfaction, the raretés will be proportional to the prices according to the equations :*

$$\frac{r_t}{p_t} = \frac{r_p}{p_p} = \frac{r_k}{p_k} = \ldots = \frac{r_a}{1} = \frac{r_b}{b} = \frac{r_c}{\pi_c} = \frac{r_d}{\pi_d} = \ldots$$

Our prices $p_t, p_p, p_k, \ldots \pi_b, \pi_c, \pi_d \ldots$ for services and products are supposed to be taken at random. We will now suppose, that there have been manufactured, say, the quantities Ω_a, Ω_b, Ω_c, $\Omega_d \ldots$ of (A), (B), (C), (D) . . ., at random, and, leaving $p_t, p_p, p_k \ldots$ as they are, let us determine the prices of (B), (C), (D) . . . by the condition that the demand for the products shall be equal to their supply, that is to the quantity manufactured. The solution of the second problem is likewise furnished us by the theory of exchange. Suppose, then, Δ_b, represented by the ordinate $\pi_b \Delta_b$ (Fig. 3), to be the total demand for (B) at the prices just supposed for services and products. We know, by the theory of exchange, that if, disregarding at first the prices of (C), (D) . . . and seeking to determine provisionally the price of (B), we cause the price to vary from zero to infinity, the demand for (B) will diminish always according to the curve $B_d B_p$. Hence, there exists a price, π'_b, corresponding to the equality of the demand for (B), with the supply Ω_b, which is $> \pi_b$, if, at the price π_b, the demand for (B) is greater than the supply, and which is $< \pi_b$, if, at the price

π_b', the supply of (B) is greater than the demand. We shall likewise find a price π_c', corresponding to the equality of the demand for (C), with the supply Ω_c, a price π_d', corresponding to the equality of the demand for (D), with the supply Ω_d, and so on. After the first experiment we proceed to a second, to a third still, and so on, until we have obtained a series of prices, π_b'', π_c'', π_d'' . . . at which the demands for (B), (C), (D) . . . will be equal to the supplies Ω_b, Ω_c, Ω_d . . . We conclude then that in the matter of production, as in the matter of exchange, *we reach the equilibrium of market of products in raising the price of those, the demand for which is greater than the supply and lowering the price of those, the supply of which is greater than the demand.*

π''_b, π''_c, π''_d . . . are thus the *selling prices* of the quantities, Ω_b, Ω_c, Ω_d . . . of (B), (C), (D) . . . But from the prices, p_t, p_p, p_k... of the services, (T), (P), (K) . . . result certain cost prices, p_b, p_c, p_d . . . of the products, (B), (C), (D) . . .* And the difference, positive or negative, between the *selling price* and the *cost price*, in the production of (B), (C), (D) . . . results in the gain or loss, $\Omega_b(\pi_b''-p_b)$, $\Omega_c(\pi_c''-p_c)$, $\Omega_d(\pi_d''-p_d)$ . . . It is now necessary to determine the manufactured quantities of (B), (C), (D) . . . by the condition that the *selling price* and *cost price* be equal, so that there may be neither gain nor

* It is true that, in order to suppose a cost price, common to all the undertakers, it is necessary to suppose that the *fixed expenses* distribute themselves among an equal quantity of products, in order to allow us to make them correspond to the *proportional expenses*; that is, it is necessary to suppose all the parties manufacturing equal quantities of products. The hypothesis is no more real than that of the absence of gain or loss, but it is as rational. If, in short, at a given point, a certain quantity of manufactured products corresponds to the absence of gain and loss, the parties in the transaction, who manufactured less, take the losses, restrain their production and finish by liquidating, those who manufactured more take the gains, develop their production and attract to themselves the business of the others; thus, owing to the distinct nature of proportional expense and fixed expense, production in free competition, after being engaged in a great number of small enterprises, tends to distribute itself among a number less great of medium enterprises, then among a small number of great enterprises, to end finally, first in a *monopoly at cost price*, then in a *monopoly at the price of maximum gain*. This statement is corroborated by the facts. But during the whole period of competition and even during the period of monopoly at cost price, it is always permissible, in order to simplify the theory, to suppose the undertakers manufacturing equal quantities of products and to make the fixed expense correspond to the proportional expense.

loss to the undertakers. This third problem is the especial problem of the theory of production, and may also be solved geometrically as follows:

Let Op_b (Fig. 4) be an abscissa representing the *cost price*, p_b. Let $O\pi''_b$ be an abscissa representing the *selling price*, π_b'', and $\pi''_b B'$ an ordinate representing the quantity Ω_b of (B), manufactured at random, and demanded at the price π_b''. If we suppose p_t, p_p, p_k, ... π_c'', π_d'' ... determined and constant, and that we may vary the price of (B), from zero to infinity, it is certain that the demand for (B) will diminish, always following a curve $B_d' B_p'$. Consequently, there exists a demand Ω_b', corresponding to a *selling price*, equal to the *cost price* p_b, which is $\gtrless \Omega_b$, according as π_b'' is $\gtrless p_b$. We might also find a demand Ω_c', corresponding to a *selling price* equal to a *cost price* p_c; a demand Ω_d', corresponding to a *selling price* equal to a *cost price* p_d, and so on. If, then, we substitute the manufactured quantities, Ω_b', Ω_c', Ω_d' ... for the manufactured quantities, Ω_b, Ω_c, Ω_d ... and sell them, according to the mechanism of rise and fall of prices described in the preceding paragraphs, we obtain new selling prices which will still be slightly different from p_b, p_c, p_d. .. Proceeding thence to a second, to a third trial, of the two experiments, and so on, we shall obtain at last certain quantities, D_b, D_c, D_d ... of (B), (C), (D) ... disposed of at *selling prices* equal to the *cost prices*, p_b, p_c, p_d ... We may, then, enunciate this important proposition for the theory of production, viz: *we arrive at the equality of the selling price of products and their cost price in productive services by augmenting the quantity of products, of which the selling price exceeds the cost price, and by diminishing the quantity of those whose cost price exceeds their selling price;* by which we see that, strictly speaking, the consideration of the expense of production determines not the *price* but the *quantity* of the products.*

* Imagine that instead of saving only himself, Robinson Crusoe had been accompanied by a hundred sailors and passengers who brought with them rice, rum, etc. If all these individuals held a market on the shore in order to exchange their

Our prices of services p_t, p_p, p_k . . . have always been determined at random. There remains to us a fourth and last problem to solve, which is to determine the way in which the quantities demanded and the quantities supplied are equal. Now, at the point where we are, there are quantities supplied of (T), (P), (K) . . . U_t, U_p, U_k . . . which are determined by the condition of maximum satisfaction, conformably to the solution of our first problem. And, in view of the quantities supplied, there are quantities demanded which are composed of two elements: first, the quantities demanded by the consumers in the way of consumable services u_t, u_p, u_k . . . which are also determined by the condition of maximum satisfaction; then, the quantities demanded by the undertakers in the way of productive services, D_t, D_p, D_k, . . . which are determined by the quantities manufactured of the products (A), (B), (C), (D) . . . the demand for which is equal to the supply, and the selling price equal to the cost price, conformably with the solution of our second and third problems. We may demonstrate exactly as in the theory of exchange, that if, everything else remaining equal, we cause p_t to vary from zero to infinity, (1), the demand for (T), $D_t + u_t$ will diminish, always following a curve $T_d T_p$ (Fig. 5); (2), the supply of (T) will, starting from zero, increase, then diminish and return to zero, following a curve QR; and that, consequently, there exists a price p'_t, at which the supply and demand of (T) are equal, which is $> p_t$, if at the price p_t, the demand for (T) is greater than the supply, and $< p_t$ if at the price p_t, the supply of (T) is greater than the demand.

commodities with each other, these would have current prices perfectly determined and entirely independent of the cost of production. This is the problem of exchange and shows how the prices depend only on the *rareté*, that is, the utility and quantity possessed of the commodities. But if, afterwards, these individuals, having found on the island the necessary productive services, proceed to manufacture the same commodities and carry their products to the market, the commodities whose selling price exceeds their cost price would multiply; those whose cost price exceeds their selling price would become rare, until the equality of selling price and cost price was established. This is the problem of production and shows how the consideration of the cost of production determines the quantity and not the price of the products.

There exists, likewise, a price p'_p, at which the supply and demand of (P) are equal, a price p'_k, at which the supply and demand of (K) are equal, and so on. After a first series of experiments with the prices, p_t, p_p, p_k . . . including, of course, the experiments in the second and third problems, we would proceed to repeat them on the prices, p_t', p_p', p_k' . . . and so on. Hence, *we arrive at the equilibrium of the market for services as in that for products, by raising the price of that for which the demand is greater than the supply, and lowering the price of that whose supply is greater than the demand.**

We must represent to ourselves all the operations as taking place simultaneously which by the requirements of the demonstration we have had to suppose occurring successively ; that is to say, in the market of products and in that of services, those who demand raise the price when the demand exceeds the supply, and those who supply lower the price when there is an excess of supply over the demand. The undertakers increase their production in case the selling price exeeeds the cost price and reduce it, on the contrary, when the cost price exceeds the selling price. And here, again, thanks to the geometric representation, we shall have an exact and complete picture of the general phenomenon of the establishment of economic equilibrium under the rule of free competition. But, nevertheless, an analytical form of expression will be necessary to a strictly scientific understanding of the matter. From this point of view, then, having defined the elements of the system or the quantities that come into play, it is necessary to distinguish those which are given and those which are unknown, to express by equations the conditions of economic equilibrium, to prove that these equations are in number just equal to the unknown quantities, to show that, by the experiments, we approximate more and more nearly the solution, and to explain the particular conditions of equilibrium so far as concerns the product (A). For all these matters, of which nothing has

* The price of the raw materials would be determined as that of productive services.

been said here, I take the liberty of referring the reader to section III of my *Eléments.*

III.

THE EXCHANGE OF SAVINGS FOR NEW CAPITALS.

In order to simplify, let us suppose for the present, the equilibrium established as regards the quantities of commodities manufactured as well as the prices of commodities and ot services, and let us neglect the changes which may be caused in this equilibrium by our investigation of the special eqiulibrium of capitalization. Let us, in the same way, neglect the cost of the redemption and insurance of the capitals.

The elements in the equilibrium of capitalization are the quantities produced of new capitals and the rate of interest whence results the prices of the capitals following the general formula $\pi = \frac{p}{i}$.

Suppose, there are produced at random, the quantities D_k, D_{k1}, D_{k11} . . . of capitals of the sorts (K), (K′), (K″) . . . and that there is a rate of interest at random, i. At that rate each man engaged in exchange determines the excess of his income over his consumption, and the total of these individual excesses forms a total excess E, which is the quantity of cash at hand to buy new capitals or the demand of the new capitals in cash at the rate of interest, i. On the other hand, at the current prices for their use, p_k, p_{k1}, p_{k11}, supposed to be determined and constant, the quantities D_k, D_{k1}, D_{k11} . . . of the capitals (K), (K′), (K″) . . . give a total income $D_k p_k + D_{k1} p_{k1} + D_{k11} p_{k11} + \ldots$, and possess a total value

$$\frac{D_k p_k + D_{k1} p_{k1} + D_{k11} p_{k11} + \ldots}{i}$$

which is the quantity of cash demanded in exchange for the new capitals or the supply of new capitals at the rate of interest i. If, by chance, the two quantities of cash are equal, the rate i will be the rate of the equilibrium of the interest,

but generally they will be unequal and it remains to render them equal.

Now, we may assume that the excess of the income over the consumption is at first nil, at a nil rate, then it multiplies and augments at a positive and increasing rate, then diminishes and returns to zero, if the rate tends to become infinitely great; that is to say, if, with a minimum saving, one may gain a very great increase in his income. In other words, the rate of interest, being an abscissa on the axis OI (Fig. 6), the excess of income over consumption will be the ordinate of a curve, successively increasing and decreasing, S T. As to the value of the new capitals it evidently increases or decreases, according as the rate of interest decreases or increases. In other words, the rate of interest being an abscissa on the axis O I, the value of the new capital may be an ordinate of a curve continually decreasing, U V. Hence, we see immediately that *it is necessary to raise the price of the new capitals by lowering the rate of interest if the demand for new capitals in cash is greater than the supply, and to lower the price of the new capitals by raising the rate of interest, if the supply of the new capitals in cash is greater than the demand.*

At this time, there are the cost prices P_k, $P_{k\prime}$, $P_{k\prime\prime}$ . . . of the new capital (K), (K′), (K″) . . . besides the selling prices π_k, $\pi_{k\prime}$, $\pi_{k\prime\prime}$ . . . The question is to reduce the selling and cost prices to the equality which generally does not exist between them. Now we may regard as established by the previous demonstrations that in augmenting or diminishing the quantity of a capital (K), we diminish or augment the *rareté* and the price of its use, and consequently the selling price of this capital, and that is to say, that the curve of the quantity in relation to the selling price is the constantly decreasing curve K_d K_p (Fig. 7). And we are equally justified in concluding that in augmenting or diminishing the quantity of the same capital (K), we augment or diminish the *rareté* and the prices of the productive services which enter into the making of the capital and con-

sequently its cost price; that is to say, that the curve of quantity in relation to the cost price is the constantly increasing curve, X Y. Hence, we see immediately and without the necessity of reproducing here the exposition of the successive approximations in regard to the quantities of capital (K), (K′), (K″) . . . that *it is necessary to augment the quantity of a new capital, whose selling price exceeds its cost price and diminish the quantity of that whose cost price exceeds its selling price.*

The equilibrium of capitalization once established, as has just been explained, we have:

$$P_k = \pi_k = \frac{p_k}{i},\ P_{k1} = \pi_{k1} = \frac{p_{k1}}{i},\ P_{k11} = \pi_{k11} = \frac{p_{k11}}{i}$$

$$\text{or}\quad \frac{p_k}{P_k} = \frac{p_{k1}}{P_{k1}} = \frac{p_{k11}}{P_{k11}} = \ldots ;$$

that is to say, the rate of interest is the same for all capital saved.

We may demonstrate geometrically in a very simple manner, at least as far as concerns capitals in consumable services, that *this identity of rate of interest is the condition of the maximum utility of new capitals.*

There are two problems of maximum utility relating to the services or use of new capitals; that connected with the distribution by an individual of his income among his different kinds of desires, and that connected with the distribution by a society of the excess of its income over its consumption among many varieties of capital. The first is solved by means of the construction which was made in the theory of exchange, and referred to at the beginning of the theory of production, involving the proportionality of the *rareté* of a species of capital to the price paid for its use, according to the equations:

$$\frac{r_k}{p_k} = \frac{r_{k1}}{p_{k1}} = \frac{r_{k11}}{p_{k11}} = \ldots$$

It will be understood, without difficulty, that the second problem would be solved by a construction exactly similar

to the former. Instead of transforming the curves of the desires for the various services of capitals by dividing the abscissas, and multiplying the ordinates by the prices for their use, p_k, p_{k1}, p_{k11} . . . we should divide the one and multiply the other by the cost prices P_k, P_{k1}, P_{k11} . . . , involving the proportionality of the *raretés* to these prices, viz:

$$\frac{r_k}{P_k} = \frac{r_{k1}}{P_{k1}} = \frac{r_{k11}}{P_{k11}} = \ldots$$

or, dividing the latter system by the former:

$$\frac{p_k}{P_k} = \frac{p_{k1}}{P_{k1}} = \frac{p_{k11}}{P_{k11}} = \ldots$$

which expresses the identity of the rates of interest of all capital.

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(Translated under the supervision of IRVING FISHER, of Yale University.)